Journal of Mathematical Analysis and Applications **261**, 113–125 (2001)
doi:10.1006/jmaa.2001.7479, available online at http://www.idealibrary.com on IDEAL®

Stability of the Shifts of Global Supported Distributions

Qiyu Sun

Department of Mathematics, National University of Singapore

Submitted by L. Debnath

Received November 7, 2000

For a tempered distribution with ℓ^1 decay, we characterize its stable shifts via its Fourier transform and via a shift-invariant space of summable sequences. Also we show that if the tempered distribution with ℓ^1 decay has stable shifts, then we can recover all distributions in V_∞, the space of all linear combinations of its shifts using bounded sequences, in a stable way using C^∞ dual functions with ℓ^1 decay at infinity. If, additionally, that tempered distribution is compactly supported, then the above C^∞ dual functions can be chosen to have exponential decay at infinity. © 2001 Academic Press

1. INTRODUCTION

Let ℓ, ℓ^p, ℓ^∞, and ℓ_0 be the spaces of all sequences on $\mathbf{Z}^d$, all p-summable sequences, all bounded sequences, and all sequences with finite support, and denote their N copies by $(\ell)^N$, $(\ell^p)^N$, $(\ell^\infty)^N$, and $(\ell_0)^N$, respectively, where $1 \le p < \infty$. The space ℓ^p, $1 \le p \le \infty$, is imbedded by the usual ℓ^p norm $\|\cdot\|_p$. It is obvious that

$$(\ell_0)^N \subset (\ell^\infty)^N \subset (\ell^p)^N \subset (\ell)^N, \qquad 1 \le p < \infty.$$

For any $W = (w(j))_{j \in \mathbf{Z}^d} \in (\ell)^N$ and $D = (d(j))_{j \in \mathbf{Z}^d} \in (\ell_0)^N$, define the action $(W, D) \mapsto D(W) = W(D)$ by

$$D(W) := \sum_{j \in \mathbf{Z}^d} d(j)^T w(-j) = \sum_{j \in \mathbf{Z}^d} w(j)^T d(-j) = W(D).$$

Here and hereafter A^T is the transpose of a vector (matrix) A.

0022-247X/01 $35.00

Denote the space of all Schwartz functions and of all compactly supported C^∞ functions by $\mathscr{S}$ and $\mathscr{D}$, respectively. Let $\langle\langle\,\cdot\,,\cdot\,\rangle\rangle$ be the action between a tempered distribution and a Schwartz function, or between a global supported distribution and a function in $\mathscr{D}$. We say that a tempered distribution F has ℓ^1 *decay* if

$$\langle\langle F(\cdot + x), h\rangle\rangle \text{ is continuous about } x \qquad \forall h \in \mathscr{S} \tag{1}$$

and if there exist positive constants C and k_0 independent of h such that

$$\|\langle\langle \mathscr{L}(F), h\rangle\rangle\|_1 \le C \sum_{|\alpha|\le k_0} \sup_{x\in\mathbf{R}^d} |D^\alpha h(x)(1+|x|)^{k_0}| \qquad \forall h \in \mathscr{S}. \tag{2}$$

Let $\mathscr{L}^p$, $1 \le p \le \infty$, be the space of all functions f such that $\sum_{j\in\mathbf{Z}^d}|f(\cdot + j)|$ is p-integrable on $[-\pi,\pi]^d$ [9]. It is easy to check that compactly supported distributions, $\mathscr{L}^p$ functions, and integrable functions on $\mathbf{R}^d$ are tempered distributions with ℓ^1 decay. Refinable distributions with smooth symbol are shown in this paper to be tempered distributions with ℓ^1 decay too (see Proposition A.7 for details).

For any tempered distribution $F = (f_1,\dots,f_N)^T$ with ℓ^1 decay, let

$$i(F) := \{\langle\langle \mathscr{L}(F),\ h\rangle\rangle\colon h \in \mathscr{D}\}$$

and

$$\mathscr{K}_\infty(F) := \left\{W \in (\ell^\infty)^N\colon W(\mathscr{L}(F)) = 0\right\},$$

where $\mathscr{L}(F)$ is an $(\ell^1)^N$-valued distribution defined by $\mathscr{L}(F) := (F(\cdot + j))_{j\in\mathbf{Z}^d}$. We remark that the space $i(F)$ is spanned by $\mathscr{L}(F)(x)$, $x \in \mathbf{R}^d$, for the case where F is continuous and compactly supported (see [18] for more properties of $i(F)$ for a compactly supported distribution F).

We say that a linear subspace of $(\ell)^N$ is *shift-invariant* if it is invariant under the *shift operators* τ_k, $k \in \mathbf{Z}^d$, where τ_k: $(w(j))_{j\in\mathbf{Z}^d} \mapsto (w(j-k))_{j\in\mathbf{Z}^d}$. Note that $\tau_k(\langle\langle \mathscr{L}(F), h\rangle\rangle) = \langle\langle \mathscr{L}(F), h(\cdot + k)\rangle\rangle$ and that $(\tau_k W)(\mathscr{L}(F)) = W(\mathscr{L}(F))(\cdot + k)$ for any $k \in \mathbf{Z}^d$, $h \in \mathscr{D}$, and $W \in (\ell^\infty)^N$. Then $i(F)$ and $\mathscr{K}_\infty(F)$ are shift-invariant. Let

$$\mathscr{I}(F) := \ell^1 \text{ closure of } i(F).$$

Note that $\langle\langle W(\mathscr{L}(F)),\ h\rangle\rangle = W(\langle\langle \mathscr{L}(F), h\rangle\rangle)$ for any $W \in (\ell^\infty)^N$ and $h \in \mathscr{D}$. Then $\mathscr{K}_\infty(F)$ is the annihilator of $\mathscr{I}(F)$ in $(l^\infty)^N$, i.e.,

$$\mathscr{K}_\infty(F) = \left\{W \in (\ell^\infty)^N\colon W(D) = 0\ \forall D \in \mathscr{I}(F)\right\}. \tag{3}$$

We say that a vector-valued tempered distribution F with ℓ^1 decay has *stable shifts* if $\mathscr{K}_\infty(F) = \{0\}$. For the stable shifts, there is a long list of publications on the characterizations and applications especially for compactly supported distributions and refinable distributions (see for instance [7, 10, 14] for compactly supported distributions, [2, 5, 6, 13, 16, 19] for compactly supported refinable distributions, and [9] for global supported functions in $\mathscr{L}^p$). From (3) it follows that

$$\text{if } \mathscr{I}(F) = (\ell^1)^N, \text{ then } F \text{ has stable shifts.} \tag{4}$$

The converse is shown in this paper to be true too.

Define the Fourier transform $\hat{f}$ by $\hat{f}(\xi) := \int_{\mathbf{R}^d} f(x) e^{-ix\xi}\, dx$ for an integrable function f and interpret the one as usual for a (vector-valued) tempered distribution. Set

$$\left[\hat{f}, \hat{g}\right](\xi) := \sum_{j \in \mathbf{Z}^d} \hat{f}(\xi + 2j\pi)\overline{\hat{g}(\xi + 2j\pi)}$$

for any measurable functions f and g with $\hat{f}\hat{g}$ being integrable. In this paper, we give the following characterization of the stable shifts of a vector-valued distribution with ℓ^1 decay.

THEOREM 1.1. *Let $F = (f_1, \dots, f_N)^T$ be a tempered distribution with ℓ^1 decay. Then the following statements are equivalent to each other.*

(i) *F has stable shifts.*

(ii) *The matrix $(\hat{F}(\xi + 2j\pi))_{j \in \mathbf{Z}^d}$ is of full rank for any $\xi \in \mathbf{R}^d$.*

(iii) *There exist $h_1, \dots, h_N \in \mathscr{D}$ such that $([\hat{f}_i, \hat{h}_{i'}](\xi))_{1 \le i,\, i' \le N}$ is invertible for any $\xi \in \mathbf{R}^d$.*

(iv) *$\mathscr{I}(F) = (\ell^1)^N$.*

For a tempered distribution $F = (f_1, \dots, f_N)^T$ with ℓ^1 decay, let

$$V_\infty(F) := \left\{D(\mathscr{L}(F)) \colon D \in (\ell^\infty)^N\right\},$$

and the *semi-convolution* $F*'\colon (\ell^\infty)^N \mapsto V_\infty(F)$ be defined by $F*'D = D(\mathscr{L}(F))$. Then the semi-convolution $F*'$ is a one-to-one correspondence between the sequence space $(\ell^\infty)^N$ and the space $V_\infty(F)$ of distributions provided that F has stable shifts. From this arises naturally the problem of how to find the inverse of the semi-convolution $F*'$. For any tempered distribution F with ℓ^1 decay, define $\mathscr{L}^*(F) = (F(\cdot - j))_{j \in \mathbf{Z}^d}$. In this paper, we show that the inverse of the semi-convolution $F*'$ can be written explicitly via a vector-valued C^∞ function with ℓ^1 decay.

THEOREM 1.2. *Let $F = (f_1, \dots, f_N)^T$ be a tempered distribution with ℓ^1 decay. If the semi-convolution $F*'\colon (\ell^\infty)^N \mapsto V_\infty(F)$ is one-to-one, then its inverse is bounded, and there is a vector-valued C^∞ function $G =$*

$(G_1, \dots, G_N)^T$ with ℓ^1 decay at infinity such that

$$(F^{*\prime})^{-1} f = \langle\langle f, \mathscr{L}^*(G)\rangle\rangle \qquad \forall\, f \in V_\infty(F).$$

Furthermore, G can be chosen to be a linear combination of the shifts of some functions $g_1, \dots, g_N \in \mathscr{D}$ using the ℓ^1 coefficients, i.e.,

$$G_i = \sum_{i'=1}^{N} \sum_{j' \in \mathbf{Z}^d} r_{ii'}(j') g_{i'}(\cdot + j'), \qquad 1 \le i \le N$$

for some sequences $(r_{ii'}(j))_{j \in \mathbf{Z}^d} \in \ell^1$, $1 \le i,\ i' \le N$.

From Theorem 1.2, when F has stable shifts, we can reconstruct all functions $f \in V_\infty(F)$ in a stable way from the average sampling $\langle\langle f, G_i(\cdot - j)\rangle\rangle$, $1 \le i \le N$, $j \in \mathbf{Z}^d$ via C^∞ functions G_i with ℓ^1 decay. If, additionally, F in Theorem 1.2 is compactly supported, then the C^∞ function G in Theorem 1.2 can be chosen to have exponential decay at infinity (see Theorem 3.4 for details).

For a vector-valued compactly supported distribution F, let

$$\mathscr{K}(F) := \left\{ W \in (\ell)^N : W(\mathscr{L}(F) = 0 \right\}.$$

We remark that $W(\mathscr{L}(F))$ is well defined for any $W \in (\ell)^N$ since $\mathscr{L}(F)$ is an $(\ell_0)^N$-valued distribution at this time. We say that F has *linear independent shifts* if $\mathscr{K}(F) = \{0\}$. Obviously if a compactly supported distribution F has linearly independent shifts, then F has stable shifts. It was proved in [1, 20] that for a compactly supported distribution F, F has linear independent shifts if and only if there exist $h_1, \dots, h_N \in \mathscr{D}$ such that the matrix $([\hat{f}_i, \hat{h}_{i'}](\xi))_{1 \le i,\, i' \le N}$ is the identity matrix for all $\xi \in \mathbf{C}^d$. So when F has linear independent shifts, we can find functions $g_1, \dots, g_N \in \mathscr{D}$ such that $(F^{*\prime})^{-1}(f) = \langle\langle f, \mathscr{L}^*(G)\rangle\rangle$ for all $f \in V_\infty(F)$, where $G = (g_1, \dots, g_N)^T$. In other words, under the linear independence assumption to F, the C^∞ function G in Theorem 1.2 can be chosen to be compactly supported.

The paper is organized as follows. The stable shifts of a tempered distribution with ℓ^1 decay is discussed in Section 2. In particular, the proofs of Theorems 1.1 and 1.2 are gathered there. Some remarks are given in Section 3, especially on the stable shifts of compactly supported distributions and the computation of the space $\mathscr{I}(F)$ of a refinable distribution F. In the Appendix, we give some basic properties of tempered distributions with ℓ^1 decay and show that a refinable distribution with smooth symbol has ℓ^1 decay.

2. PROOFS

Let $\mathscr{F}(D) := \sum_{j\in\mathbf{Z}^d} d(j)e^{-ij\cdot}$ be the Fourier series of a sequence $D \in \ell^1$, and $\mathscr{F}^{-1}(f)$ be the Fourier sequence of a 2π periodic distribution f. Denote the space of all 2π periodic functions f with Fourier series in ℓ^1 by $\mathscr{W}$, and define $\|f\|_{\mathscr{W}} := \|\mathscr{F}^{-1}(f)\|_1$. Any function in the space $\mathscr{W}$ is said to belong to the Wiener class. For any function $R \in \mathscr{W}$ such that $R(\xi) \neq 0$ for all $\xi \in \mathbf{R}^d$, the Wiener theorem says that $R^{-1} \in \mathscr{W}$ too.

Proof of Theorem 1.1. We divide the proof into the following steps: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i). Note that it follows from (4) that (iv) implies (i). Then it remains to prove the rest of the implications.

(i) ⇒ (ii). On the contrary, suppose that there exist $\alpha \in \mathbf{C}^N \setminus \{0\}$ and $\xi_0 \in \mathbf{R}^d$ such that $\alpha^T\hat{F}(\xi_0 + 2j\pi) = 0$ for all $j \in \mathbf{Z}^d$. This together with (A.1) and Proposition A.6 implies that

$$\left\langle\!\!\left\langle \sum_{j\in\mathbf{Z}^d} e^{ij\xi_0}\alpha^T F(\cdot - j), h\right\rangle\!\!\right\rangle = \sum_{j\in\mathbf{Z}^d} e^{ij\xi_0}\langle\!\langle \alpha^T F(\cdot - j), h\rangle\!\rangle$$

$$= \left[\alpha^T\hat{F}, \hat{h}\right](\xi_0) = 0 \qquad \forall h \in \mathscr{S}.$$

Therefore $(e^{ij\xi_0}\alpha)_{j\in\mathbf{Z}^d} \in \mathscr{K}_\infty(F)$, which is a contradiction.

(ii) ⇒ (iii). By the assumption, there exists a finite set $K(\xi) \subset \mathbf{Z}^d$ for any $\xi \in [-\pi, \pi]^d$ such that $\operatorname{rank}(\hat{F}(\xi + 2k\pi))_{k\in K(\xi)} = N$. Let $\varphi \in \mathscr{D}$ be supported in $[-\pi/2, \pi/2]^d$ and take value one at the origin, and define $h_{\xi,k} \in \mathscr{S}$, $k \in K(\xi)$, by $\hat{h}_{\xi,k}(\eta) = \varphi(\eta - \xi - k)$. Noting that $[\hat{F}, \hat{h}_{\xi_0,k}](\xi_0) = F(\xi_0 + 2k\pi)$ for any $\xi_0 \in [-\pi, \pi]^d$ and $k \in K(\xi_0)$, then by the compactness of $[-\pi, \pi]^d$ and continuity of $\hat{F}$ by Proposition A.6, there exist a finite set ξ_λ, $\lambda \in \Lambda$, and $h_{\lambda,k_\lambda} \in \mathscr{S}$, $k_\lambda \in K(\xi_\lambda)$, such that $([\hat{F}, \hat{h}_{\lambda,k_\lambda}](\xi))_{k_\lambda\in K(\xi_\lambda),\,\lambda\in\Lambda}$ has full rank N for any $\xi \in [-\pi, \pi]^d$. This, together with (2) and the density of functions in $\mathscr{D}$ in $\mathscr{S}$, leads to

$$\operatorname{rank}\left(\left[\hat{F}, \hat{h}_i\right](\xi)\right)_{1\le i\le r} = N \qquad \forall\, \xi \in [-\pi, \pi]^d \tag{5}$$

for some functions $h_1, \ldots, h_r \in \mathscr{D}$. Note that $[\hat{F}, h_i]$, $1 \le i \le r$, belong to the Wiener class. Then by (5), there exists a $r \times N$ matrix $B(\xi)$ with entries in the Wiener class such that

$$\left(\left[\hat{F}, \hat{h}_i\right](\xi)\right)_{1\le i\le r} B(\xi) = I_N \qquad \forall \xi \in [-\pi, \pi]^d. \tag{6}$$

By the density of trigonometric polynomials in the Wiener class, for some sufficiently small constant $\delta > 0$ chosen later, there exists a $r \times N$ matrix $\tilde{B}(\xi)$ with trigonometric polynomial entries $b_{si'}(\xi)$, $1 \le s \le r$, $1 \le i' \le N$, such that

$$\|B(\xi) - \tilde{B}(\xi)\|_{\mathscr{W}} \leq \delta. \tag{7}$$

Then it follows from (6) and (7) that

$$\left\| \left(\left[\hat{f}_i, \sum_{s=1}^{r} \overline{b_{si'}} \hat{h}_s \right](\xi) \right)_{1 \leq i,\, i' \leq N} - I_N \right\|_{\mathscr{W}} \leq C\delta \tag{8}$$

for some positive constant C independent of δ. Then $([\hat{f}_i, \hat{H}_{i'}](\xi))_{1 \leq i, i' \leq N}$ is invertible for any $\xi \in [-\pi, \pi]^d$ by defining $H_1, \ldots, H_N$ by $\hat{H}_{i'}(\xi) = \sum_{s=1}^r \overline{b_{si'}(\xi)} \hat{h}_s(\xi)$, $1 \leq i' \leq N$, and choosing δ sufficient small.

(iii) $\Rightarrow$ (iv). Let v_i, $1 \leq i \leq N$, be vectors with ith component one and other components zero. Then it suffices to prove that for any $\varepsilon > 0$, there exists $H_1^\epsilon, \ldots, H_N^\epsilon \in \mathscr{D}$ such that

$$\left\| \left[\hat{F}, \hat{H}_i^\epsilon \right](\xi) - v_i \right\|_{\mathscr{W}} \leq \epsilon, \qquad 1 \leq i \leq N. \tag{9}$$

Let $h_1, \ldots, h_N \in \mathscr{D}$ be as in (iii). Then $([\hat{f}_i, \hat{h}_{i'}](\xi))$ is invertible for any $\xi \in \mathbf{R}^d$. By the density of trigonometric polynomials in the Wiener space, there exists an $N \times N$ matrix $B^\epsilon(\xi) = (b_{ii'}^\epsilon(\xi))_{1 \leq i,\, i' \leq N}$ with trigonometric polynomial entries such that

$$\left\| \left(\left[\hat{f}_i, \hat{h}_{i'} \right](\xi) \right)_{1 \leq i,\, i' \leq N} B^\epsilon(\xi) - I_N \right\|_{\mathscr{W}} \leq \epsilon. \tag{10}$$

Define $H_1^\epsilon, \ldots, H_N^\epsilon$ by $\hat{H}_i^\epsilon(\xi) = \sum_{i'=1}^N \hat{h}_{i'}(\xi) \overline{b_{i'i}^\epsilon(\xi)}$, $1 \leq i \leq N$. Then (9) follows from (10). ∎

Proof of Theorem 1.2. Let $h_1, \ldots, h_N \in \mathscr{D}$ be as in (iii) of Theorem 1.1. Then $B(\xi) = ([\hat{f}_i, \hat{h}_{i'}](\xi))_{1 \leq i, i' \leq N}$ is invertible for any $\xi \in \mathbf{R}^d$. Recall that all entries of $B(\xi)$ are in the Wiener class by (2) and (A.1). Then $(B(\xi))^{-1} = (\mathscr{F}(R_{ii'})(\xi))_{1 \leq i, i' \leq N}$ for some sequences $R_{ii'} = (r_{ii'}(j))_{j \in \mathbf{Z}^d} \in \ell^1$ by the Wiener theorem. Therefore we obtain

$$\sum_{i''=1}^{N} \sum_{j' \in \mathbf{Z}^d} \langle\langle f_i(\cdot + j), h_{i''}(\cdot + j') \rangle\rangle r_{i''i'}(j') = \delta_{ii'} \delta_{j0}. \tag{11}$$

Let $G_i = \sum_{i'=1}^N \sum_{j' \in \mathbf{Z}^d} r_{i'i}(j') h_{i'}(\cdot + j')$, $1 \leq i \leq N$, and $G = (G_1, \ldots, G_N)^T$. Then it follows from (11) that

$$D = \langle\langle D(\mathscr{L}(F)), \mathscr{L}^*(G) \rangle\rangle \qquad \forall D \in (\ell^\infty)^N.$$

This proves that $(F^{*\prime})^{-1}(f) = \langle\langle f, \mathscr{L}^*(G) \rangle\rangle$ for all $f \in V_\infty(F)$. ∎

3. SOME REMARKS

In this section, we discuss the stable shifts of a compactly supported distribution, and the computation of the space $\mathscr{I}(F)$ for a refinable distribution F.

3.1. *The Stable Shifts of Compactly Supported Distributions*

Define the convolution $D * W$ of two summable sequences $D = (d(j))_{j \in \mathbf{Z}^d}$ and $W = (w(j))_{j \in \mathbf{Z}^d}$ by $D * W := (\sum_{j' \in \mathbf{Z}^d} d(j')w(j - j'))_{j \in \mathbf{Z}^d}$. Then $\mathscr{F}(D * W) = \mathscr{F}(D)\mathscr{F}(W)$ for any D and $W \in \ell^1$. For any compactly supported distribution $F = (f_1, \dots, f_N)^T$, let

$$i_S(F) = \Big\{ D \in (\ell_0)^N : D * R \in i(F) \text{ for some } R \in \ell_0 \\ \text{with } \mathscr{F}(R)(\xi) \neq 0\ \forall \xi \in \mathbf{R}^d \Big\}.$$

It is easy to check that $i_S(F)$ is shift-invariant, and $i(F) \subset i_S(F) \subset \mathscr{I}(F) \cap (\ell_0)^N$. For the stable shifts of a compactly supported distribution, we have the following result corresponding to Theorem 1.1, except the subspace $\mathscr{I}(F)$ of $(\ell^1)^N$ is replaced by the subspace $i_S(F)$ of $(\ell_0)^N$.

THEOREM 3.3. *Let $F = (f_1, \dots, f_N)^T$ be a compactly supported distribution on $\mathbf{R}^d$. Then the following statements are equivalent to each other.*

(i) *F has stable shifts.*

(ii) *The matrix $(\hat{F}(\xi + 2j\pi))_{j \in \mathbf{Z}^d}$ is of full rank for any $\xi \in \mathbf{R}^d$.*

(iii) *There exist $h_1, \dots, h_N \in \mathscr{D}$, and a trigonometric polynomial $\alpha(\xi)$ with $\alpha(\xi) \neq 0$ for any $\mathbf{R}^d$ such that $([\hat{f}_i, \hat{h}_{i'}](\xi))_{1 \le i, i' \le N} = \alpha(\xi) I_N$ for any $\xi \in \mathbf{R}^d$, where I_N is the $N \times N$ identity matrix.*

(iv) $i_S(F) = (\ell_0)^N$.

We remark that the equivalence between (i) and (ii) in Theorem 3.3 was proved in [14] for $N = 1$ and [10] for $N \ge 1$.

THEOREM 3.4. *Let $F = (f_1, \dots, f_N)^T$ be a compact supported distribution. If the semi-convolution $F*'$: $(\ell^\infty)^N \mapsto V_\infty(F)$ is one-to-one, then its inverse is bounded, and there exists a vector-valued C^∞ function $G = (G_1, \dots, G_N)^T$ with exponential decay at infinity such that $(F*')^{-1} f = \langle\langle f, \mathscr{L}^*(G) \rangle\rangle$ for all $f \in V_\infty(F)$. Furthermore, G can be chosen to be a linear combination of the shifts of some functions $g_1, \dots, g_N \in \mathscr{D}$ using sequences with exponential decay, i.e.,*

$$G_i = \sum_{j \in \mathbf{Z}^d} r(j) g_i(\cdot + j), \qquad 1 \le i \le N,$$

and there exist positive constants C *and* λ_0 *independent of* j *such that* $|r(j)| \le Ce^{-\lambda_0|j|}$ *for all* $j \in \mathbf{Z}^d$.

We omit the detail of the proofs of Theorems 3.3 and 3.4 since we can use the same procedure to prove Theorems 1.1 and 1.2 except using the fact that the inverse of a trigonometric polynomial $R(\xi)$ with $R(\xi) \neq 0$ for all $\xi \in \mathbf{R}^d$ has exponential decay instead of using the Wiener theorem.

3.2. *Refinable Distributions*

A tempered distribution $F = (f_1, \dots, f_N)^T$ is said to be *refinable* if it has a continuous Fourier transform and satisfies a *refinement equation*

$$F = \sum_{j \in \mathbf{Z}^d} c(j) F(2 \cdot - j), \tag{12}$$

where $c(j)$, $j \in \mathbf{Z}^d$, are $N \times N$ matrices such that $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$. The existence of distributional solutions of the refinement equation (2) is well studied especially for compactly supported solutions (see for instance [3, 4, 8, 12, 17, 21]). A square matrix is said to satisfy *Condition E* if one is its spectral radius, its only eigenvalue on the unit circle, and its simple eigenvalue. Let $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j) e^{-ij\xi}$, which is known as the *symbol* $H(\xi)$ of the refinement equation (12), or of the refinable distribution F. In the Appendix of this paper, we show that if $H(\xi) \in C^{d+1}$ and $H(0)$ satisfies Condition E, then the nonzero distributional solution of the refinement equation (12) has ℓ^1 decay (see Proposition A.7 for details).

For the $N \times N$ matrices $c(j)$, $j \in \mathbf{Z}^d$, with $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$, define

$$B_k = \big(c(2i - i' + k)\big)_{i, i' \in \mathbf{Z}^d}, \qquad k \in \mathbf{Z}^d. \tag{13}$$

It is easy to check that B_k, $k \in \mathbf{Z}^d$, are bounded operators on $(\ell^1)^N$. For the shift-invariant space $\mathscr{I}(F)$, we have

THEOREM 3.5. *Let* F *be a nonzero distributional solution of the refinement equation* (12) *with continuous Fourier transform*, $c(j)$, $j \in \mathbf{Z}^d$, *be the family of* $N \times N$ *matrices in* (12) *such that* $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$, *and let* Φ *be a compactly supported* $C^{k_0(F)}$ *function such that*

$$\lim_{n \to \infty} \sum_{|\alpha| \le k_0(F)} \sup_{x \in \mathbf{R}^d} |D^\alpha (h - h_n)(x)(1 + |x|)^{k_0}| = 0 \qquad \forall h \in \mathscr{S}, \tag{14}$$

where $k_0(F)$ *is the minimal integer* k_0 *such that* (2) *holds for* F, *and* h_n, $n \ge 1$, *are finite linear combinations of* $\Phi(2^n \cdot - j)$, $j \in \mathbf{Z}^d$. *Assume that* $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j) e^{-ij\xi} \in C^{d+1}$ *and that* $H(0)$ *satisfies Condition E. Then* $\mathscr{I}(F)$ *is the minimal closed subspace of* $(\ell^1)^N$ *which is invariant under* B_k, $k \in \mathbf{Z}^d$, *and which contains the initial sequence* $\langle\langle \mathscr{L}(F), \Phi \rangle\rangle$.

Similar invariant spaces of refinable distributions can be found in [7, 18].

Proof of Theorem 3.5. By Proposition A.7, F is a tempered distribution with ℓ^1 decay. Hence $\mathscr{I}(F)$ is well defined and shift-invariant. Let V be the minimal closed linear subspace of $(\ell^1)^N$ which is invariant under B_k, $k \in \mathbf{Z}^d$, and which contains the sequence $\langle\langle \mathscr{L}(F), \Phi\rangle\rangle$. Note that for any $h \in \mathscr{D}$ and $k \in \mathbf{Z}^d$, $B_k D = 2^d \langle\langle \mathscr{L}(F), h(2\cdot - k)\rangle\rangle$ by (12). Then it suffices to prove

$$\langle\langle \mathscr{L}(F), \Phi\rangle\rangle \in \mathscr{I}(F), \tag{15}$$

and for any $h \in \mathscr{D}$,

$$\langle\langle \mathscr{L}(F), h\rangle\rangle \in V. \tag{16}$$

Let $g \in \mathscr{D}$ with $\hat{g}(0) = 1$, and let $g_n * \Phi$, $n \geq 1$, be the convolution between g_n and Φ, where $g_n(x) = n^d g(x/n)$. Then $g_n * \Phi \in \mathscr{D}$ and are supported in a compact set independent of $n \geq 1$. By $\Phi \in C^{k_0(F)}$.

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}^d} |D^\alpha(g_n * \Phi - \Phi)(x)| = \lim_{n\to\infty} \|g_n * (D^\alpha\Phi) - D^\alpha\Phi\|_\infty = 0$$

for any $|\alpha| \leq k_0(F)$. Therefore $\langle\langle \mathscr{L}(F), g_n * \Phi\rangle\rangle$ converges to $\langle\langle \mathscr{L}(F), \Phi\rangle\rangle$ in $(\ell^1)^N$ by (2). This, together with $\langle\langle \mathscr{L}(F), g_n * \Phi\rangle\rangle \in i(F)$ and closedness of $\mathscr{I}(F)$ in $(\ell^1)^N$ topology, implies that $\langle\langle \mathscr{L}(F), \Phi\rangle\rangle \in \mathscr{I}(F)$. This proves (15).

By the assumption on h_n, $\langle\langle \mathscr{L}(F), h_n\rangle\rangle \in V$ since for any $k \in \mathbf{Z}^d$,

$$\langle\langle \mathscr{L}(F), \Phi(2^n\cdot - k)\rangle\rangle = 2^{-nd} B_k B_0^{n-1} \langle\langle \mathscr{L}(F), \Phi\rangle\rangle \in V.$$

Hence $\langle\langle \mathscr{L}(F), h\rangle\rangle \in V$ by (2), (14), and closedness of V. ∎

Let us make a remark to compute the initial sequence $\langle\langle \mathscr{L}(F), \Phi\rangle\rangle$ when Φ is chosen to be refinable too. Assume that Φ is a compactly supported $C^{k_0(F)}$ function and satisfies (14) and the following refinement equation

$$\Phi = \sum_{j\in\mathbf{Z}^d} d(j)\Phi(2\cdot - j), \tag{17}$$

where the sequence $(d(j))_{j\in\mathbf{Z}^d}$ has finite support. Set $\Psi(x) = \langle\langle F(\cdot + x), \Phi\rangle\rangle$. Then Ψ is continuous by (2), and

$$\begin{aligned}\Psi(x) &= 2^{-d} \sum_{j, j'\in\mathbf{Z}^d} c(j)d(j')\langle\langle F(2x - j + \cdot), \Phi(\cdot - j')\rangle\rangle \\ &= \sum_{j\in\mathbf{Z}^d} \left(2^{-d} \sum_{j'\in\mathbf{Z}^d} c(j + j')d(j')\right)\Psi(2x - j)\end{aligned} \tag{18}$$

by (12) and (17). Define an operator B on $(\ell^1)^N$ by

$$B = \left(2^{-d} \sum_{j' \in \mathbf{Z}^d} c(2i - i' + j')d(j')\right)_{i, i' \in \mathbf{Z}^d}.$$

Then $(\Psi(j))_{j \in \mathbf{Z}^d}$ is an eigenvalue of B with eigenvalue one by (17) and the continuity of Φ. Hence $\langle\langle \mathscr{L}(F), \Phi \rangle\rangle = \mathscr{L}(\Psi)(0)$ can be computed if the operator B on $(\ell^1)^N$ has one as its simple eigenvalue. This gives a way to compute $\langle\langle \mathscr{L}(F), G \rangle\rangle$ through finding the eigenvector of the operator B with eigenvalue one.

APPENDIX: DISTRIBUTIONS WITH ℓ^1 DECAY

In this appendix, we give some basic properties of tempered distributions with ℓ^1 decay and show the distributional solution of a refinement equation with smooth symbol has ℓ^1 decay.

PROPOSITION A.6. *Let F be a vector-valued tempered distribution on $\mathbf{R}^d$ with ℓ^1 decay. Then*

(i) *$\hat{F}$ is continuous.*

(ii) *$\hat{F}$ has polynomial increase at infinity; i.e., there exists a polynomial $Q(\xi)$ such that $|\hat{F}(\xi)| \le Q(\xi)$ for all $\xi \in \mathbf{R}^d$.*

(iii) *$\sum_{j \in \mathbf{Z}^d} d(j)^T F(\cdot - j)$ converges in distributional sense for any sequence $(d(j))_{j \in \mathbf{Z}^d} \in (\ell^\infty)^N$.*

(iv) *For any $g \in \mathscr{D}$, $[\hat{F}, \hat{g}]$ is well defined and continuous, and*

$$\left[\hat{F}, \hat{g}\right](\xi) = \sum_{j \in \mathbf{Z}^d} \langle\langle F(\cdot + j), g \rangle\rangle e^{-ij\xi}. \tag{A.1}$$

Proof. For any Schwartz function h, set $\|h\|_{\mathscr{D}^{k_0}} = \sum_{|\alpha| \le k_0} \|D^\alpha h(x)(1 + |x|)^{k_0}\|_\infty$ and $h_t = h(\cdot - t)$ for any $t \in \mathbf{R}^d$. Then there exists a positive constant C independent of h and t such that $\|h_t\|_{\mathscr{D}^{k_0}} \le C\|h\|_{\mathscr{D}^{k_0}}$ for all $|t| \le 1$ and $h \in \mathscr{S}$. This together with (2) leads to

$$\sum_{j \in \mathbf{Z}^d} |h * F(t + j)| = \sum_{j \in \mathbf{Z}^d} |h_t * F(j)| \le C\|h\|_{\mathscr{D}^{k_0}}.$$

Then by integrating over $[0,1]^d$ at the both sides of the above inequality, we get $h * F \in L^1$ and

$$\|h * F\|_1 \le C\|h\|_{\mathscr{D}^{k_0}}. \tag{A.2}$$

Therefore $\hat{h}\hat{F}$ is bounded and continuous by the Riemann Lemma, and the continuity of $\hat{F}$ follows.

Let $\varphi \in \mathscr{D}$ with $\varphi(\xi) = 1$ on the unit disk and define h_N, $N \geq 1$, by $\hat{h}_N(\xi) = (1 + |\xi|)^{-N_1}\varphi(\xi/N)$, where $N_0 \geq k_0 + d + 2$. Then there exist positive constants C_1 and C_2 independent of $N \geq 1$ such that

$$\|h_N\|_{\mathscr{D}^{k_0}} \leq C_1 \sum_{|\beta|, |\alpha| \leq k_0} \|D^{\beta}\big(\hat{h}_N(\xi)\xi^{\alpha}\big)\|_1 \leq C_2 \qquad \forall\ N \geq 1.$$

This together with (A.2) leads to

$$\sup_{|\xi| \leq N} (1 + |\xi|)^{-N_1} |\hat{F}(\xi)| \leq \|\hat{h}_N \hat{F}\|_{\infty} \leq \|h_N * F\|_1 \leq C,$$

where C is a positive constant independent of N. This proves (ii).

For any $D = (d(j))_{j \in \mathbf{Z}^d} \in (\ell^{\infty})^N$ and the Schwartz function h, define

$$D_F(h) = \lim_{K \to \infty} \sum_{|j| \leq K} d(j) \langle\langle F(\cdot - j), h \rangle\rangle.$$

Then for any Schwartz function h, $D_F(h)$ is well-defined and

$$|D_F(h)| \leq \|D\|_{\infty} \|\langle\langle \mathscr{L}(F), h \rangle\rangle\|_1 \leq C\|D\|_{\infty}\|h\|_{\mathscr{D}^{k_0}}$$

by (2). This shows that $h \mapsto D_F(h)$ is a continuous linear form on the space of all Schwartz functions. Denote that continuous form by $\sum_{j \in \mathbf{Z}^d} d(j)^T F(\cdot - j) = D(\mathscr{L}(F))$. By the above procedure, $\sum_{|j| \leq K} d(j)^T F(\cdot - j)$ converges to $D(\mathscr{L}(F))$ in the distributional sense.

By the assumptions on F and g, $\hat{F}\overline{\hat{g}}$ is continuous and integrable. Thus $[\hat{F}, \hat{g}]$ is well defined and continuous. By direct computation, we have

$$\begin{aligned} \langle\langle F(\cdot + j), g \rangle\rangle &= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ij\xi} \hat{F}(\xi) \overline{g(\xi)}\, d\xi \\ &= (2\pi)^{-d} \int_{[-\pi, \pi]^d} e^{ij\xi} \big[\hat{F}, \hat{g}\big](\xi)\, d\xi \qquad \forall\ j \in \mathbf{Z}^d. \end{aligned}$$

Hence (A.1) follows from the uniqueness theorem about the Fourier sequences of 2π periodic integrable functions. ∎

PROPOSITION A.7. *Let $F = (f_1, \dots, f_N)^T$ be a tempered distribution, have a continuous Fourier transform, and satisfy the refinement equation $F = \sum_{j \in \mathbf{Z}^d} c(j) F(2 \cdot - j)$, where $\sum_{j \in \mathbf{Z}^d} \|c(j)\| < \infty$. Define $H(\xi) = 2^{-d} \sum_{j \in \mathbf{Z}^d} c(j) e^{-ij\xi}$. Assume that $H(\xi) \in C^{d+1}$ and that $H(0)$ satisfies Condition E. Then F is a tempered distribution with ℓ^1 decay.*

To prove Proposition A.7, we need the following estimate about the limit of $H(\xi/2)H(\xi/4)\cdots H(\xi/2^n)$. We omit the detail of the proof here since it can be proved by the same method as in [3, 8].

LEMMA A.8. *Let $H(\xi)$ be a matrix-valued 2π periodic function on $\mathbf{R}^d$ and set $\hat{F}_n(\xi) = H(\xi/2)\cdots H(\xi/2^n)$, $n \geq 0$. Assume that $H(\xi) \in C^{d+1}$ and that $H(0)$ satisfies Condition E. Then for any $|\alpha| \leq d+1$, $D^\alpha F_n(\xi)$ converges uniformly to some continuous function on any compact set. Furthermore there exist positive constants C and B independent of n and ξ such that*

$$\|D^\alpha \hat{F}_n(\xi)\| \leq C(1+|\xi|)^B \tag{A.3}$$

for all $|\alpha| \leq r$, $n \geq 1$, and $\xi \in \mathbf{R}^d$.

Proof of Proposition A.7. Let $\hat{F}_n(\xi) = H(\xi/2)H(\xi/4)\cdots H(\xi/2^n)$, $n \geq 1$, be as in Lemma A.8. Then $F_n(\xi)$ converges, and its limit $M(\xi)$ is continuous and dominated by a polynomial, which implies that $M(\xi)$ is a tempered distribution. Denote the inverse Fourier transform of $M(\xi)\hat{F}(0)$ by G. Then G is a tempered distribution too. Now it remains to prove that $G = F$ and that G is a tempered distribution with ℓ^1 decay.

Define $H_K(\xi) = \sum_{|j|\leq K} c(j)e^{-ij\xi}$. Then from the definition of F, $H_K\hat{F}$ converges to $\hat{F}(2\,\cdot)$ in the distributional sense, since $\sum_{|j|\leq K} c(j)F(2\cdot - j)$ converges to F in the distributional sense. On the other hand, $H_N(\xi)\hat{F}(\xi)$ converges to $H(\xi)\hat{F}(\xi)$ uniformly on any compact set, and hence converges in the distributional sense. Therefore $\hat{F}(\xi) = H(\xi/2)\hat{F}(\xi/2)$. Using the above formula for n times leads to $\hat{F}(\xi) = H(\xi/2)\cdots H(\xi/2^n)\hat{F}(\xi/2^n)$. Then letting n tend to infinity and using the continuity of $\hat{F}$ and Lemma A.8, we obtain $\hat{F}(\xi) = M(\xi)\hat{F}(0)$. This proves $F = G$.

By (A.3), $\hat{G}\hat{h}$ is integrable for any Schwartz function h, hence $\langle\langle G(\cdot + x), h\rangle\rangle$ is continuous about x. This proves (1) for G. Still by (A.3), there exist positive constants C and B independent of ξ and α such that

$$|D^\alpha \hat{G}(\xi)| \leq C(1+|\xi|)^B \tag{A.4}$$

for all $|\alpha| \leq d+1$ and $\xi \in \mathbf{R}^d$. Therefore for any Schwartz function h,

$$\begin{aligned}
&\sum_{j\in\mathbf{Z}^d}\left|\int_{\mathbf{R}^d}\hat{G}(\xi)\hat{h}(\xi)c^{-ij\xi}\,d\xi\right| \\
&\quad\leq C_1\sum_{j\in\mathbf{Z}^d}(1+|j|)^{-d-1}\sup_{\xi\in\mathbf{R}^d}\sum_{|\gamma|\leq d+1}\left|D^\gamma\big(\hat{G}\hat{h}(\xi)\big)\right|(\mathbf{1}+|\xi|)^{d+1} \\
&\quad\leq C_2\sup_{\xi\in\mathbf{R}^d}\sum_{|\gamma|\leq d+1}\left|D^\gamma\hat{h}(\xi)\right|(1+|\xi|)^{B+d+1} < \infty,
\end{aligned}$$

where C_1, C_2 are positive constants independent of h, and B is the constant in (A.4). This proves (2) for G. ∎

REFERENCES

1. A. Ben-Artzi and A. Ron, On the integer translates of a compactly supported function: dual bases and linear projectors, *SIAM J. Math. Anal.* **21** (1990), 1550–1562.
2. A. Cohen and Q. Sun, An arithmetic characterization of the conjugate quadrature filters associated to orthonormal wavelet bases, *SIAM J. Math. Anal.* **24** (1993), 1355–1360.
3. D. Colella and C. Heil, Matrix refinement equations: Existence and uniqueness, *J. Fourier Anal. Appl.* **2** (1996), 363–377.
4. I. Daubechies, "Ten Lectures on Wavelets," SIAM, Philadelphia, 1992.
5. T. A. Hogan, Stability and independence of the shifts of finitely many refinable functions, *J. Fourier Anal. Appl.* **3** (1997), 757–774.
6. T. A. Hogan, Stability and independence for multivariate refinable distributions, *J. Approx. Theory* **98** (1999), 248–270.
7. T. A. Hogan and R. Q. Jia, Dependency relations among the shifts of a multivariate refinable distribution, *Constr. Approx.* **17** (2001), 19–37.
8. D. Huang, Q. Sun, and W. Wang, Multiresolution generated by several scaling functions, *Adv. in Math. (China)* **26**, No. 2 (1997), 165–180.
9. R. Q. Jia and C. A. Micchelli, Using the refinement equation for the construction of pre-wavelets. II. Powers of two, *in* "Curve and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), pp. 209–246, Academic Press, New York, 1991.
10. R. Q. Jin and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.* **36** (1992), 69–75.
11. R. Q. Jia and J.-Z. Wang, Stability and linear independence associated with wavelet decompositions, *Proc. Amer. Math. Soc.* **117** (1993), 1115–1124.
12. Q. Jiang and Z. Shen, On existence and weak stability of matrix refinable functions, *Constr. Approx.* **15** (1999), 337–353.
13. W. Lawton, Necessary and sufficient conditions for constructing orthonormal wavelet bases, *J. Math. Phys.* **32** (1991), 57–61.
14. A. Ron, A necessary and sufficient condition for the linear independence of the integer translates of a compactly supported distribution, *Constr. Approx.* **5** (1989), 297–308.
15. A. Ron, Introduction to shift-invariant spaces. I. Linear independence, preprint, 1999.
16. Z. Shen, Refinable function vectors, *SIAM J. Math. Anal.* **29** (1998), 239–250.
17. Q. Sun, Compactly supported distributional solutions of nonstationary nonhomogeneous refinement equation, *Acta Math. Sinica* **17** (2001), 1–14.
18. Q. Sun, Linear dependence of the shifts of vector-valued compactly supported distributions, preprint, 2000.
19. J.-Z. Wang, Stability and linear independence associated with scaling vectors, *SIAM J. Math. Anal.* **29** (1998), 1140–1156.
20. K. Zhao, Global linear independence and finitely supported dual basis, *SIAM J. Math. Anal.* **23** (1992), 1352–1355.
21. D.-X. Zhou, Existence of multiple refinable distributions, *Michigan Math. J.* **44** (1997), 317–329.